Algorithms Seminar 2002–2004,
F. Chyzak (ed.), INRIA, (2005), pp. 71–74.

Available online at the URL
http://algo.inria.fr/seminars/.

Efficient Computation of a Class of Continued Fraction Constants

Loïck Lhote
GREYC, Université de Caen (France)

November 17, 2003

Summary by Julien Clément

Abstract

There are numerous instances where mathematical constants do not admit a closed form. It is then of great interest to compute them, possibly in an efficient way. So the question is: does there exist an algorithm that computes the first d-digits of the constants and if so, what is the complexity in the number of arithmetic operations? We recall that a constant is said to be *polynomial-time computable* if its first d digits can be obtained with $O(d^r)$ arithmetic operations. Here we consider a particular class of constants arising in the field of the dynamical analysis of algorithms and dynamical systems. The constants to compute are of a "spectral" nature since they are closely related to the spectrum of some transfer operators.

1. Dynamical Systems and Constants

Dynamical analysis of algorithms was introduced by Brigitte Vallée [10] and gives a general framework to study complex and realistic systems. Roughly speaking, the idea is to track the execution of an algorithm through trajectories in its associated dynamical system. The techniques are based upon functional analysis and generating operators. This approach has been successfully applied to euclidean algorithms and yields very precise (and new) results. Viewing the dynamical system as a way to produce symbols, this framework allows also to study data structures built upon words like digital trees and tries [1, 9].

In the general and basic setting [6, 7], we are interested in complete dynamical systems $(\mathcal{I}, T)$ formed with an interval $\mathcal{I}$ and a map $T : \mathcal{I} \to \mathcal{I}$ which is piecewise surjective and of class $\mathcal{C}^2$. We denote by $\mathcal{G}$ the set of the inverse branches of T; then, $\mathcal{G}^k$ is the set of the inverse branches of T^k. It is known that contraction properties of the inverse branches are essential to obtain "good" properties on the dynamical system. Usually, what is needed is the existence of a disk D which is strictly mapped inside itself by all the inverse branches $h \in \mathcal{G}$ of the system (i.e., $h(\overline{D}) \subset \overline{D}$).

If f_0 is an initial density on $\mathcal{I}$, repeated applications of the map T modify the density and the successive densities $f_1, f_2, \ldots, f_n, \ldots$ describes the global evolution of the system at time $t = 0, 1, 2 \ldots, n, \ldots$. The operator $\mathbf{G}$ such that $f_1 = \mathbf{G}[f_0]$ and more generally $f_n = \mathbf{G}[f_{n-1}] = \mathbf{G}^n[f_0]$ for all n is called the density transformer. It is defined as

$$\mathbf{G}[f](z) = \sum_{h \in \mathcal{G}} |h'(z)| \; f \circ h(z)$$

It acts on the functional space $A_\infty(D)$ for some convenient disk D where $\mathcal{I} \subset D$ and

$$A_\infty(D) := \{f : D \to \mathbb{C}; f \text{ analytic on } D \text{ and continuous on } \overline{D}\}.$$

A perturbation of the density transformer, the transfer operator $\mathbf{G}_s$, defined as

$$\mathbf{G}_s[f](z) = \sum_{h \in \mathcal{G}} |h'(z)|^s \; f \circ h(z)$$

involves a new (complex) parameter s. It extends the density transformer since $\mathbf{G}_1 = \mathbf{G}$ and plays a crucial rôle in the analysis of rational trajectories. It acts on $A_\infty(D)$ as soon as $\Re s > 1/2$. Remark that its iterate $\mathbf{G}^n$ of order n involves the set $\mathcal{G}^n$ of the inverse branches of depth n,

$$\mathbf{G}_s^n[f](z) = \sum_{h \in \mathcal{G}^n} |h'(z)|^s \; f \circ h(z).$$

Example. Let us consider the Euclidean dynamical system related to the Gauss map to give a more precise view. Every real number $x \in]0, 1]$ admits a continued fraction expansion of the form

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p + \dots}}}},$$

where the m_i form a sequence of positive integers. Ordinary continued fraction expansions of real numbers are the result of an iterative process which constitutes the continuous counterpart of the standard Euclidean division algorithm. They can be viewed as trajectories of a specific dynamical system relative to the Gauss map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{for } x \neq 0, \qquad T(0) = 0,$$

(here, $\lfloor x \rfloor$ is the integer part of x). The set $\mathcal{G}$ of the inverse branches of T is $\mathcal{G} := \{h : x \mapsto \frac{1}{m+x} \text{ for } m \geq 1\}$ and the transfer operator is

$$\mathbf{G}_s[f](z) = \sum_{h \in \mathcal{G}} |h'(z)|^s \; f \circ h(z) = \sum_{m \geq 1} \frac{1}{(m+z)^{2s}} \, f\left(\frac{1}{m+z}\right).$$

One of the main and most useful property of these transfer operators is that, under a "contracting condition," for real s the operator $\mathbf{G}_s$ has a unique dominant eigenvalue $\lambda(s)$, positive and isolated from the remainder of the spectrum by a spectral gap.

The constants we wish to compute are of three kinds and are all related to this dominant eigenvalue $\lambda(s)$;

1. Evaluate $\lambda(r)$ for some r real. For example $\lambda(2)$ is related to the coincidence probability and appears in the analysis of the height of digital trees.
2. Evaluate $\lambda'(r)$ or $\lambda''(r)$ for some r real. For instance, $-\lambda'(1)$ is the entropy of the dynamical system and plays a central rôle. The quantity $\lambda''(1)$ intervenes in the expression of the variance for the average number of steps of the classical Euclidean algorithm.
3. In the context of computing a local expansion of the quasi-inverse $(\mathrm{Id} - \mathbf{G}_s)^{-1}$, compute r such that $\lambda(r) = 1$. Also this is related to the computation of Hausdorff dimension of Cantor-like sets associated with incomplete dynamical systems.

2. The Principles of the Algorithm

For any function $f \in A_\infty(D)$, the Taylor expansions at $x_0 \in \mathcal{I}$ of f and $\mathbf{G}[f]$ exist and the operator $\mathbf{G}$ can be viewed as an infinite matrix $\mathbf{M} := (M_{i,j})$ with $0 \le i, j < \infty$ and

$$M_{i,j} = \text{ the coefficient of } (z - x_0)^i \text{ in } \mathbf{G}[(Z - x_0)^j](z).$$

The truncated matrix $\mathbf{M}_n := (M_{i,j})_{0 \le i,j \le n}$ is the matrix of order $n+1$ which describes the action of a "truncated" operator on the space $\mathcal{P}_n$ formed with polynomials of degree at most n. More precisely, the truncated matrix $\mathbf{M}_n$ represents the truncated operator $\pi_n \circ \mathbf{G}|_{\mathcal{P}_n}$ where π_n is the projection on $\mathcal{P}_n$ which associates to a function f its Taylor expansion of order n at x_0 i.e.,

$$\pi_n[f](z) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \; (z - x_0)^k.$$

Note that the operator $\pi_n \circ \mathbf{G}$ and the matrix $\mathbf{M}_n$ have the same spectrum.

In the case of the Euclidean dynamical system, Daudé, Flajolet, and Vallée introduced in [2] a method for computing (a finite part of) the spectrum of transfer operators, which they further used in [3, 8]. Their method, the so–called DFV–method, has three main steps which we describe in an informal way (see Figure 1).

(i) Compute the truncated matrix $\mathbf{M}_n$ relative to the operator $\mathbf{G}$.

(ii) Compute the spectrum $\operatorname{Sp} \mathbf{M}_n$ of matrix $\mathbf{M}_n$, i.e., the set of its eigenvalues,

$$\operatorname{Sp} \mathbf{M}_n := \{ \lambda_n^{(i)} : 0 \le i \le n \}.$$

(iii) Relate the set $\operatorname{Sp} \mathbf{M}_n$ with a (finite) part of $\operatorname{Sp} \mathbf{G}$.

$$\begin{array}{ccc} \text{Operator } \mathbf{G} & \xrightarrow{\text{Hard}} & \text{Eigenvalue } \lambda \\ {\scriptstyle\text{Truncate}}\big\downarrow & & \big\uparrow {\scriptstyle |\lambda_n - \lambda| \le K\theta^n} \\ \text{Matrix } \mathbf{M}_n & \xrightarrow[\text{Easy}]{} & \text{Eigenvalue } \lambda_n \end{array}$$

FIGURE 1. The DFV-method for computing eigenvalue approximates.

In the case when the transfer operator has a unique dominant eigenvalue λ, isolated from the remainder of the spectrum, one can expect that it is the same for $\mathbf{M}_n$, with a dominant eigenvalue λ_n. Moreover, the authors of [2] observed that the sequence λ_n seems to converge to λ (when the truncation degree n tends to ∞), with exponential speed. They conjectured the following:

There exist n_0, K, θ such that, for any $n \ge n_0$, one has $|\lambda_n - \lambda| \le K\theta^n$.

The main result here is that this can be proved in a very general setting, i.e., as soon as the dynamical system considered is strongly contracting and not just contracting.

A complete dynamical system is said to be *strongly contracting* if the set $\mathcal{G}$ of inverse branches fulfills the supplementary conditions: For any subset of the inverse branches $\mathcal{A} \subset \mathcal{G}$, there exist x_0 and two open disks of center x_0 D_S and D_L with $D_S \subsetneqq D_L$ and $\mathcal{I} \subset D_L$, such that any $h \in \mathcal{A}$ is an element of $A_\infty(D_L)$ which strictly maps D_L inside D_S, i.e., $h(\overline{D}_L) \subset \overline{D}_S$.

Many dynamical systems relative to the Euclidean algorithms belong to this strongly contracting setting (and even to an extra-contracting setting where another disk comes between D_L and D_S and the constraint is put on the inverse branches of an iterate at a certain order of the map T – see [5] for a precise definition).

3. Effective Computation

One has under the strongly contracting condition that the truncated operator $\pi_n \circ \mathbf{G}$ converges in norm to $\mathbf{G}$. However an effective algorithm to solve the problem involves a circle $C = C(\lambda, r)$ (with center λ and radius $r > 0$) that isolates the dominant eigenvalue λ from the remainder of the spectrum and the quantity α_C defined by

$$\alpha_C(\mathbf{G}) := \sup_{z \in C} \|(\mathbf{G} - z\mathrm{Id})^{-1}\|.$$

In general it is an open problem to compute this quantity $\alpha_C(\mathbf{G})$ even if the dynamical system is strongly contracting. There is an exception when the operator satisfies a normality property on some functional space. Fortunately this is the case for the usual Euclidean dynamical system (but on an Hardy space and not $A_\infty(D)$). So this gives an effective algorithm to compute polynomially the constants. Finally evaluation of constants related to derivatives of the function $\lambda(s)$ is done thanks to Taylor expansion.

As a conclusion, let us just state the numerical value of the *Hensley constant* [4] with seven (proved) digits

$$\gamma_H = 2\frac{\lambda''(1) - \lambda'(1)}{\lambda'(1)^3} \approx 0.5160524\ldots$$

This constant appears in the variance of the average number of divisions P_n in the Euclid algorithm on a pair (u, v) such that $0 \le u \le v \le n$ and which is $\mathbf{Var}\, P_n \sim \gamma_H \log n$. The reader is referred to [5] for numerical values of other constants like the *Gauss–Kuz'min–Wirsing constant* and the *Hausdorff dimension* of the Cantor set $\mathcal{R}_{\{1,2\}}$.

Bibliography

[1] Clément (Julien), Flajolet (Philippe), and Vallée (Brigitte). – Dynamical sources in information theory: a general analysis of trie structures. *Algorithmica*, vol. 29, n° 1-2, 2001, pp. 307–369. – Average-case analysis of algorithms (Princeton, NJ, 1998).

[2] Daudé (Hervé), Flajolet (Philippe), and Vallée (Brigitte). – An average-case analysis of the Gaussian algorithm for lattice reduction. *Combinatorics, Probability and Computing*, vol. 6, 1997, pp. 397–433.

[3] Flajolet (Philippe) and Vallée (Brigitte). – Continued fraction algorithms, functional operators and structure constants. In *Proceedings of the Canadian Mathematical Society*, vol. Constructive, Experimental, and Nonlinear Analysis (in the honour of Jonathan Borwein). – 2000.

[4] Hensley (Doug). – The number of steps in the Euclidean algorithm. *Journal of Number Theory*, vol. 49, n° 2, 1994, pp. 142–182.

[5] Lhote (Loïck). – Computation of a class of continued fraction constants. In *Proceedings of the Sixth Workshop on Algorithm Engineering and Experiments and the First Workshop on Analytic Algorithmics and Combinatorics, New Orleans, LA, USA, January 10, 2004*, pp. 199–210. – 2004.

[6] Mayer (Dieter H.). – Spectral properties of certain composition operators arising in statistical mechanics. *Communications in Mathematical Physics*, vol. 68, 1979, pp. 1–8.

[7] Mayer (Dieter H.), T. Bedford (M. Keane) and Series (C.) (editors). – *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Chapter Continued fractions and related transformations, pp. 175–222. – Oxford University Press, 1991.

[8] Vallée (Brigitte). – Dynamique des fractions continues à contraintes périodiques. *Journal of Number Theory*, vol. 72, n° 2, 1998, pp. 183–235.

[9] Vallée (Brigitte). – Dynamical sources in information theory: fundamental intervals and word prefixes. *Algorithmica*, vol. 29, n° 1-2, 2001, pp. 262–306. – Average-case analysis of algorithms (Princeton, NJ, 1998).

[10] Vallée (Brigitte). – Dynamical analysis of a class of Euclidean algorithms. *Theoretical Computer Science*, vol. 297, n° 1-3, 2003, pp. 447–486. – Latin American theoretical informatics (Punta del Este, 2000).